ON THE STABILITY OF A RIGID BODY WITH A FIXED POINT

(OB USTOICHIVOSTI TVERDOGO TELA S ZAKREPLENNOI TOCHKOI)

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We consider the problem of stability of motion, in the sense of Liapunov, of a heavy rigid body with a fixed point. The body moves on the surface of the Earth's sphere and its center of gravity C is not located on one of the principal axes of inertia at the point of support. We use the integral of the energy of the system in order to obtain the sufficient condition of stability of the unperturbed motion of the system.

1. Under the assumption that the center of gravity is located on one of the principal axes of inertia at the point of support the problem has been solved in [1]. Without this assumption the problem becomes considerably more complicated and it becomes much more difficult to determine the position of equilibrium of the system and the form of stability conditions.

Let $Ox_1y_1z_1$ and $Oxyz$ be right handed coordinate systems with their origins at the point of support. The x_1-axis coincides with the direction of the velocity vector of the point of support, the z_1-axis is along the Earth's radius vertically up [2]. The axes x, y, z are fixed in the body, the dirction of z-axis is such that the center of gravity of the body is on this axis.

We could have selected a different orientation of the x,y,z axes, for example we could have made them coincide with the three principal axes of inertia of the body at the point of support, and the orientation of the vector OC in this system could have been determined by its x,y,x components. However, such a choice of coordinates does not make the problem any simpler.

The equations of motion of the system projected on the x,y,z axes have the form

$$\frac{dK_x}{dt} + qK_z - rK_y = (F_g - m\omega v)\, l_z\psi_2 + mvl_z\Omega\vartheta_2 + ml_z\frac{dv}{dt}\theta_2$$

$$\frac{dK_y}{dt} + rK_x - pK_z = -(F_g - m\omega v)\, l_z\psi_1 - mvl_z\Omega\vartheta_1 - ml_z\frac{dv}{dt}\theta_1 \qquad (1.1)$$

$$\frac{dK_z}{dt} + pK_y - qK_x = 0 \qquad \left(\omega = \frac{v}{R}\right)$$

$$K_x = Ap - Fq - Er, \qquad K_y = Bq - Dr - Fp, \qquad K_z = Cr - Ep - Dq \qquad (1.2)$$

Here A, B, C, F, E, D are the respective axial and centrifugal moments of inertia of the system; p,q,r are the x,y,z components of the angular velocity of the xyz trihedron; F_g is the force of gravitation; m is the mass of the body; v is the velocity of the point of support; R is the

Earth's radius; l_z is the z component of the vector OC ; Ω is the angular velocity of the trihedron $x_1 y_1 z_1$ about the z_1-axis. This system in which v and Ω are constants permits the energy integral analogous to the one shown in [3]. This last has the form

$$V \cong [{}^1\!/_2 p - (\Omega\psi_1 + \omega\vartheta_1)] K_x + [{}^1\!/_2 q - (\Omega\psi_2 + \omega\vartheta_2)] K_y + [{}^1\!/_2 r - (\Omega\psi_3 + \omega\vartheta_3)] K_z + \\ + (F_g - m\omega v) l_z\psi_3 + mvl_z\Omega\vartheta_3 = h \tag{1.3}$$

2. The integral (1.3) can be used for obtaining sufficient conditions of stability. The equilibrium occurs when the coordinates assume the following values

$$\alpha = \alpha_0, \qquad \beta = \beta_0, \ \gamma = \gamma_0 \tag{2.1}$$

and α_0, β_0, γ_0 satisfy Equations

$$\begin{gathered}[qK_z - rK_y - (F_g - m\omega v) l_z\psi_2 - mvl_z\Omega\vartheta_2]_0 = 0 \\ [rK_x - pK_z + (F_g - m\omega v) l_z\psi_1 + mvl_z\Omega\vartheta_1]_0 = 0 \\ (pK_y - qK_x)_0 = 0\end{gathered} \tag{2.2}$$

The subscript $_0$ in (2.2) and later on will indicate that the corresponding functions are taken at the values (2.1). Assuming that the motion determined by Equations (2.1) is unperturbed, we obtain, analogously to [3], that this motion is stable in the Liapunov sence if all the principal diagonal minors of the matrices

$$B = \| b_{ij} \| \quad (3\times3) \qquad C = \| c_{ij} \| \quad (3\times3) \tag{2.3}$$

$$b_{11} = [{}^1\!/_2 (A\psi_1^2 + B\psi_2^2 + C\psi_3^2) - (F\psi_1\psi_2 + E\psi_1\psi_3 + D\psi_2\psi_3)]_0 \tag{2.4}$$

$$b_{22} = [{}^1\!/_2 (A \cos^2\gamma + C \sin^2\gamma) - E \cos\gamma \sin\gamma]_0, \qquad b_{33} = {}^1\!/_2 B$$

$$b_{12} = b_{21} = {}^1\!/_2 [(A\psi_1 - F\psi_2 - E\psi_3) \cos\gamma + (C\psi_3 - E\psi_1 - D\psi_2) \sin\gamma]_0$$

$$b_{13} = b_{31} = {}^1\!/_2 (B\psi_2 - F\psi_1 - D\psi_3)_0, \qquad b_{23} = b_{32} = -{}^1\!/_2 (F \cos\gamma + D \sin\gamma)_0$$

$$c_{11} = {}^1\!/_2\omega \{A (p\vartheta_1 - \omega\theta_1^2) + B (q\vartheta_2 - \omega\theta_2^2) + C (r\vartheta_3 - \omega\theta_3^2) + \\ + F [2\omega\theta_1\theta_2 - (p\vartheta_2 + q\vartheta_1)] + E [2\omega\theta_1\theta_3 - (p\vartheta_3 + r\vartheta_1)] + \\ + D[2\omega\theta_2\theta_3 - (q\vartheta_3 + r\vartheta_2)] - mR\Omega l_z\vartheta_3\}_0$$

$$c_{22} = {}^1\!/_2 \{[(B - C)(q^2 - r^2) - Fpq - Epr - 4Dqr - \\ - (F_g - m\omega v) l_z\psi_3 - mv\Omega l_z\vartheta_3] \cos^2\gamma + [(A - B)(p^2 - q^2) - \\ - 4Fpq - Epr - Dqr] \sin^2\gamma + [(2B - A - C) pr + 3q (Fr + Dp) + \\ + E (p^2 + r^2 - 2q^2) + (F_g - m\omega v) l_z\psi_1 + mv\Omega l_z\vartheta_1] \cos\gamma \sin\gamma\}_0$$

$$c_{33} = {}^1\!/_2 [(A - C)(p^2 - r^2) - Fpq - 4Epr - Dqr - (F_g - m\omega v) l_z\psi_3 - mv\Omega l_z\vartheta_3]_0$$

$$c_{12} = c_{21} = {}^1\!/_2\omega \{[(C - B)(q\theta_3 + r\theta_2) + F (p\theta_3 + r\theta_1) - E (p\theta_2 + q\theta_1) + \\ + 2D (r\theta_3 - q\theta_2) - mR\Omega l_z\theta_2] \cos\gamma + [(B - A)(p\theta_2 + q\theta_1) + \\ + 2F (q\theta_2 - p\theta_1) + E (q\theta_3 + r\theta_2) - D (p\theta_3 + r\theta_1)] \sin\gamma\}_0$$

$$c_{13} = c_{31} = {}^1\!/_2 \omega [(A - C)(p\theta_3 + r\theta_1) - F (q\theta_3 + r\theta_2) + 2E (p\theta_1 - r\theta_3) + \\ + D (p\theta_2 + q\theta_1) + mR\Omega l_z\theta_1]_0$$

$$c_{23} = c_{32} = {}^1\!/_2 \{[(C - A) pq + F (q^2 - r^2) + r (2Eq + Dp)] \cos\gamma + [(A - C) qr + \\ + p (Fr + 2Eq) + D (q^2 - p^2) + (F_g - m\omega v) l_z\psi_2 + mv\Omega l_z\vartheta_2] \sin\gamma\}_0$$

are positive.

Let us mention that Equations (2.2) have the solution $\alpha_0 = 0$, $\beta_0 = 0$, $\gamma_0 = 0$ if

$$(C - B)\omega\Omega + D (\Omega^2 - \omega^2) - mvl_z\Omega = 0, \qquad F\omega + E\Omega = 0 \tag{2.5}$$

In this case the sufficent conditions of stability assume the simple form

$$\begin{gathered}C > 0, \quad AC - E^2 > 0, \quad ABC - 2FED - AD^2 - BE^2 - CF^2 > 0 \\ (B - A)\omega - D\Omega > 0\end{gathered} \tag{2.6}$$

(2.6) cont.

$$(B-A)\,\omega-D\Omega]\,[(C-B)\,(\Omega^2-\omega^2)-4D\omega\Omega-(F_g-m\omega v)\,l_z]-\omega\,(F\Omega-E\omega)^2>0$$
$$-(F_g-m\omega v-mR\Omega^2)\,l_z>0$$

3. We shall assume that the x, y, z axes are the principal axes of inertia of the body at the point of support, $l_z<0$ and the conditions [4] (unperturbed physical pendulum)

$$C=0,\qquad A=B=-ml_zR \tag{3.1}$$

are satisfied.

Then the relations (2.5) will be also satisfied and the condition of stability of the system will be [5 to 7] the well known inequality

$$F_g-m\omega v-mR\Omega^2>0 \tag{3.2}$$

4. Having the choice in the selection of the x, y, z axes we can direct them along the principal axes of inertia at the point of support and determine the orientation of the vector OC by its components l_x, l_y, l_z. In certain cases such a choice of axes can be obtained either directly as in the preceding cases, or by using the previously obtained results and performing in them a suitable transformation of coordinates.

The energy integral of the system has the form

$$V\equiv[{}^1/_2\,p-(\Omega\psi_1+\omega\vartheta_1)]\,Ap+[{}^1/_2\,q-(\Omega\psi_2+\omega\vartheta_2)]\,Bq+[{}^1/_2\,r-(\Omega\psi_3+\omega\vartheta_3)]\,Cr+$$
$$+(F_g-m\omega v)(l_x\psi_1+l_y\psi_2+l_z\psi_3)+mv\Omega\,(l_x\vartheta_1+l_y\vartheta_2+l_z\vartheta_3)=h \tag{4.1}$$

The unperturbed motion

$$\alpha=\alpha_0,\qquad\beta=\beta_0,\qquad\gamma=\gamma_0 \tag{4.2}$$

where α_0, β_0, γ_0 satisfy Equations

$$\begin{aligned}&[(C-B)\,qr+(F_g-m\omega v)(l_y\psi_3-l_z\psi_2)+mv\Omega\,(l_y\vartheta_3-l_z\vartheta_2)]_0=0\\&[(A-C)\,rp+(F_g-m\omega v)(l_z\psi_1-l_x\psi_3)+mv\Omega\,(l_z\vartheta_1-l_x\vartheta_3)]_0=0\\&[(B-A)\,pq+(F_g-m\omega v)(l_x\psi_2-l_y\psi_1)+mv\Omega\,(l_x\vartheta_2-l_y\vartheta_1)]_0=0\end{aligned} \tag{4.3}$$

will be stable in the Liapunov sense if all the principal diagonal minors of the matrix

$$C=\|c_{ij}\|\,(3\times3) \tag{4.4}$$

$$c_{11}={}^1/_2\,\omega\,[A\,(p\vartheta_1-\omega\theta_1^2)+B\,(q\vartheta_2-\omega\theta_2^2)+C\,(r\vartheta_3-\omega\theta_3^2)-$$
$$-mR\Omega\,(l_x\vartheta_1+l_y\vartheta_2+l_z\vartheta_3)]_0 \tag{4.5}$$

$$c_{22}={}^1/_2\,\{[(B-C)(q^2-r^2)-(F_g-m\omega v)(l_y\psi_2+l_z\psi_3)-mv\Omega\,(l_y\vartheta_2+l_z\vartheta_3)]\cos^2\gamma+$$
$$+[(A-B)(p^2-q^2)-(F_g-m\omega v)(l_x\psi_1+l_y\psi_2)-mv\Omega\,(l_x\vartheta_1+l_y\vartheta_2)]\sin^2\gamma+$$
$$+[(2B-A-C)\,pr+(F_g-m\omega v)(l_x\psi_3+l_z\psi_1)+mv\Omega\,(l_x\vartheta_3+l_z\vartheta_1)]\cos\gamma\sin\gamma\}_0$$

$$c_{33}={}^1/_2\,[(A-C)(p^2-r^2)-(F_g-m\omega v)(l_x\psi_1+l_z\psi_3)-mv\Omega\,(l_x\vartheta_1+l_z\vartheta_3)]_0$$

$$c_{12}=c_{21}={}^1/_2\,\omega\,\{[(C-B)\,(q\theta_3+r\theta_2)+mR\Omega\,(l_y\theta_3-l_z\theta_2)]\cos\gamma+[(B-A)(p\theta_2+q\theta_1)+$$
$$+mR\Omega\,(l_x\theta_2-l_y\theta_1)]\sin\gamma\}_0$$

$$c_{13}=c_{31}={}^1/_2\,\omega\,[(A-C)(p\theta_3+r\theta_1)+mR\Omega\,(l_z\theta_1-l_x\theta_3)]_0$$

$$c_{23}=c_{32}={}^1/_2\,\{[(C-A)\,pq+(F_g-m\omega v)\,l_x\psi_2+mv\Omega l_x\vartheta_2]\cos\gamma+$$
$$+[(A-C)\,qr+(F_g-m\omega v)\,l_z\psi_2+mv\Omega l_z\vartheta_2]\sin\gamma\}_0$$

are positive.

We shall consider now a special case of the problem. The center of gravity of the body is located in the yz plane ($l_x=0$). The values $\alpha=0$, $\beta=\beta_0$, $\gamma=\theta$ correspond to the position of equilibrium of the system.

The coordinate β_0 satisfies Equation

$$\{(C-B)\left[{}^1/_2(\Omega^2-\omega^2)\sin 2\beta+\Omega\omega\cos 2\beta\right]+(F_g-m\omega v)(l_y\cos\beta-l_z\sin\beta)-\\ -mv\Omega(l_z\cos\beta+l_y\sin\beta)\}_0=0 \qquad (4.6)$$

The sufficient conditions of stability of the unperturbed motion are

$$c_{11}>0, \quad c_{22}>0, \quad c_{11}c_{33}-c_{13}^2>0 \qquad (4.7)$$

$$c_{11}={}^1/_2\,\omega\left[-A\omega+B\omega_1\cos\beta-C\omega_2\sin\beta+mR\Omega(l_z\sin\beta-l_y\cos\beta)\right]_0 \qquad (4.8)$$

$$c_{22}={}^1/_2\left[(B-C)(\omega_1^2-\omega_2^2)-(F_g-m\omega v)(l_y\sin\beta+l_z\cos\beta)+mv\Omega(l_z\sin\beta-l_y\cos\beta)\right]_0$$

$$c_{33}={}^1/_2\left[(C-A)\,\omega_2^2-(F_g-m\omega v)\,l_z\cos\beta+mv\Omega l_z\sin\beta\right]_0$$

$$c_{13}={}^1/_2\,\omega\left[(A-C)\,\omega_2+mR\Omega l_z\right]_0, \qquad \omega_1=\omega\cos\beta+\Omega\sin\beta, \qquad \omega_2=-\omega\sin\beta+\Omega\cos\beta$$

Equation (4.6) has the solution $\beta_0=0$, if l_y, l_z satisfy the relation

$$(C-B)\,\Omega\omega=-(F_g-m\omega v)\,l_y+mv\Omega l_z \qquad (4.9)$$

In this case the inequalities (4.7) take on the simple form

$$(B-A)\,\omega-mR\Omega l_y>0, \quad (B-C)(\omega^2-\Omega^2)-(F_g-m\omega v)\,l_z-mv\Omega l_y>0 \qquad (4.10)$$
$$[(B-A)\,\omega-mR\Omega l_y]\,[(C-A)\,\Omega^2-(F_g-m\omega v)\,l_z]-\\ -\omega\,[(A-B)\,\Omega+(R/v)(F_g-m\omega v)\,l_y]^2>0$$

When $l_y=0$, $l_z<0$ then from (4.7) and (4.10) we obtain the conditions of stability of a spherical pendulum, which have been previously obtained in [1].

The condition (4.9), in particular, is satisfied if $l_y=0$ and if the relations (3.1) are satisfied. In this last case the sufficient condition of stability is (3.2).

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